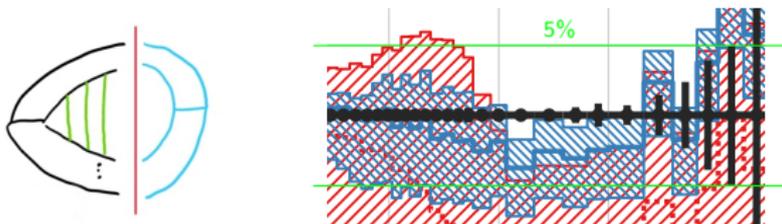


Amplitudes and Precision at Hadron Colliders



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Snowmass CSS, University of Washington, Seattle, 7/22/2022

Based on the Snowmass White Paper [[arXiv:2204.04200](https://arxiv.org/abs/2204.04200)]
With **Andreas von Manteuffel** and **Tobias Neumann**

References From the Snowmass Process

- Snowmass reports

- **Energy Frontier**: Narain, Reina, Tricoli, (& contributors)
- **EF05-07 – QCD Report**: Begel, Höche, Lin, Mukherjee, Nadolsky, Royon, Schmitt, (& contributors)
- **TF04 – Scattering Amplitudes and their Applications**: Bern, Trnka, (& contributors)
- **TF06 – Theory Techniques for Precision Physics** : Boughezal, Ligeti, (& contributors)
- **TF07 – Theory of Collider Phenomena**: Maltoni, Su, Thaler, (& contributors)

- Snowmass white papers

- **Computational challenges for multi-loop collider phenomenology**: FFC, von Manteuffel, Neumann [[arXiv:2204.04200](https://arxiv.org/abs/2204.04200)]
- **The Path forward to N³LO**: Caola, Chen, Duhr, Liu, Mistlberger, Petriello, Vita, Weinzierl [[arXiv:2203.06730](https://arxiv.org/abs/2203.06730)]
- See also: Special funcs [[arXiv:2203.07088](https://arxiv.org/abs/2203.07088)], Coll factorization [[arXiv:2207.06507](https://arxiv.org/abs/2207.06507)]

And references therein!

References From the Snowmass Process

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Apologies in advance for not covering all the impressive related activity!

Introduction

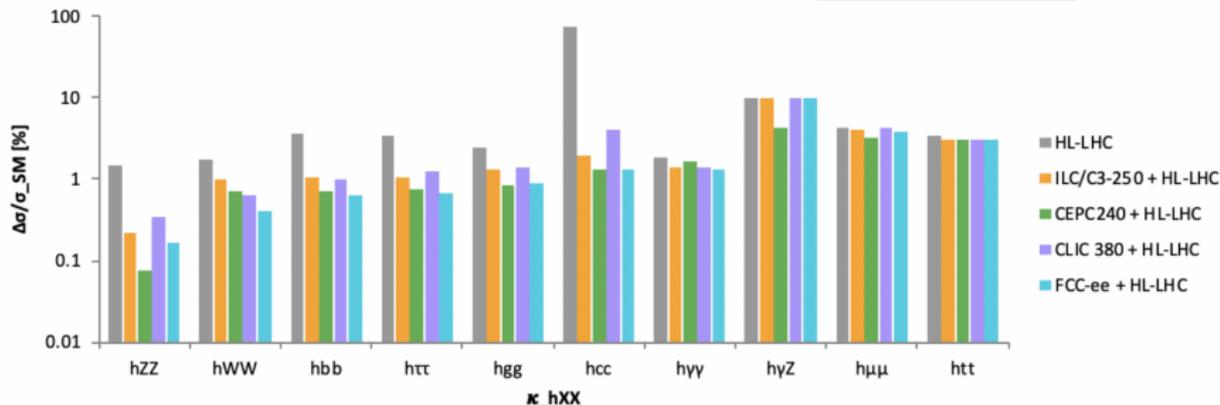
State-of-the-art and future needs

A touch on techniques

Our survey & outlook

Snowmass Projections for Higgs Couplings

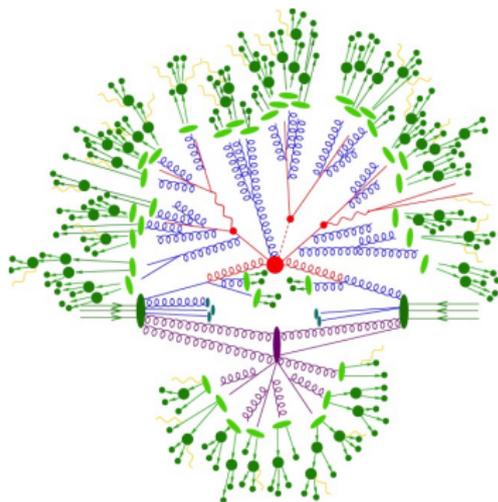
Energy Frontier Report



HL-LHC can achieve $\mathcal{O}(\text{few } \%)$ errors for Higgs coupling measurements

Critical input from multi-scale theory predictions, typically in processes involving 3 (or more) FS objects

Hadron Collider Event Simulation



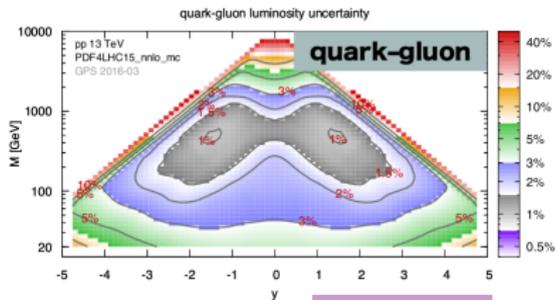
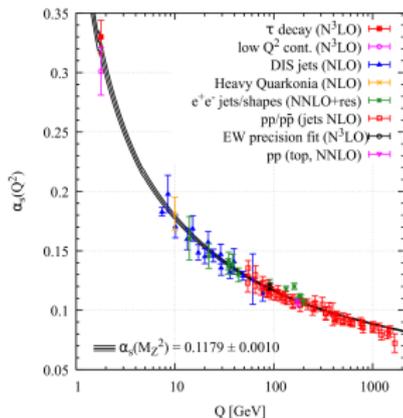
- Factorization
- Hard scattering
- Parton evolution
- Simulation underlying event and hadronization
- Particle decays and radiation

$$\sigma_{h_1 h_2 \rightarrow H} = \sum_{a,b} \int dx_a dx_b f_{a/h_1}(x_a, \mu_F) f_{b/h_2}(x_b, \mu_F) \hat{\sigma}_{ab \rightarrow H+X}(\mu_F, \mu_R)$$

Snowmass contribution: Sterman [[arXiv:2207.06507](https://arxiv.org/abs/2207.06507)]

Uncertainties in Perturbative Predictions

Parametric: determination of model's parameters
couplings, PDFs, masses

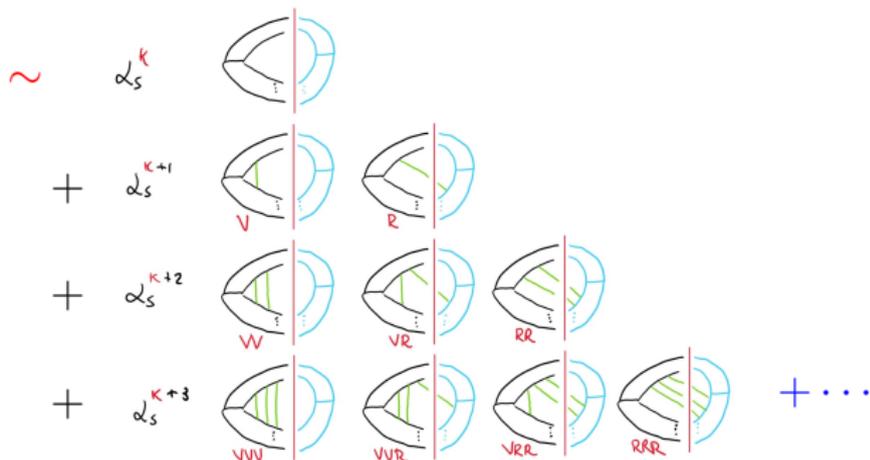


Perturbative truncation

$$\hat{\sigma}_{ab \rightarrow H} = \alpha_s^\kappa \left(\sigma_{LO} + \alpha_s \sigma_{NLO} + \alpha_s^2 \sigma_{NNLO} + \alpha_s^3 \sigma_{N^3LO} + \dots \right)$$

Truncation and Underlying Amplitudes

$$\sigma_{h_1 h_2 \rightarrow H} = \alpha_s^k (\sigma_{\text{LO}} + \alpha_s \sigma_{\text{NLO}} + \alpha_s^2 \sigma_{\text{NNLO}} + \alpha_s^3 \sigma_{\text{N}^3\text{LO}} + \dots)$$

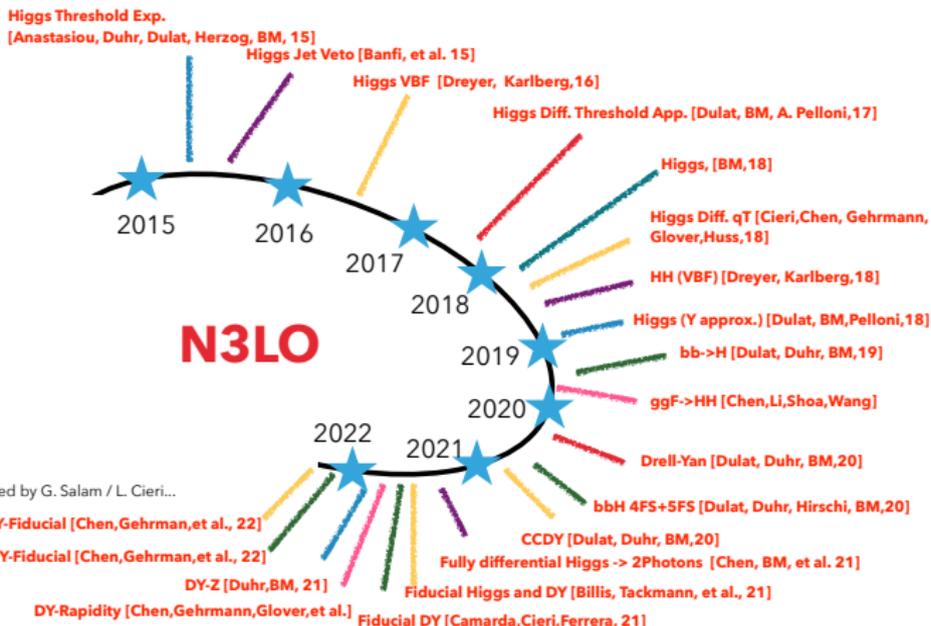


A **myriad** of amplitudes are required for precision calculations

N³LO Progress in Time

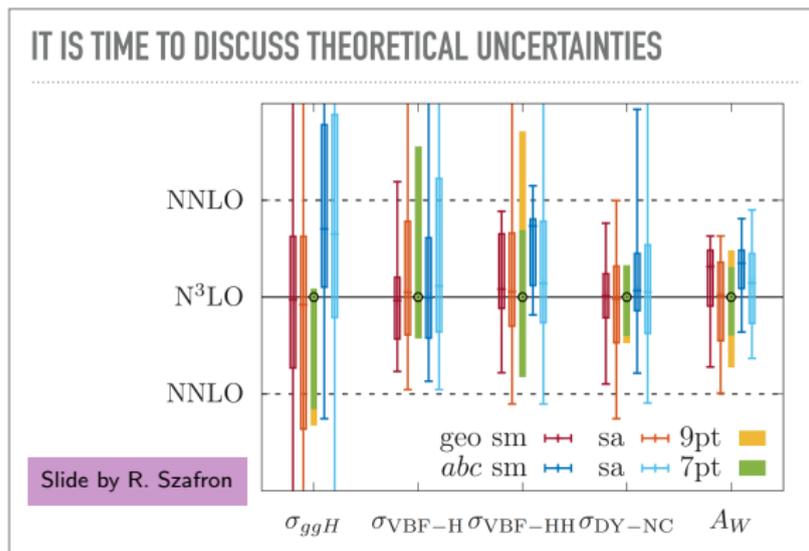
N³LO AT THE LHC OVER TIME

Slide by B. Mistlberger



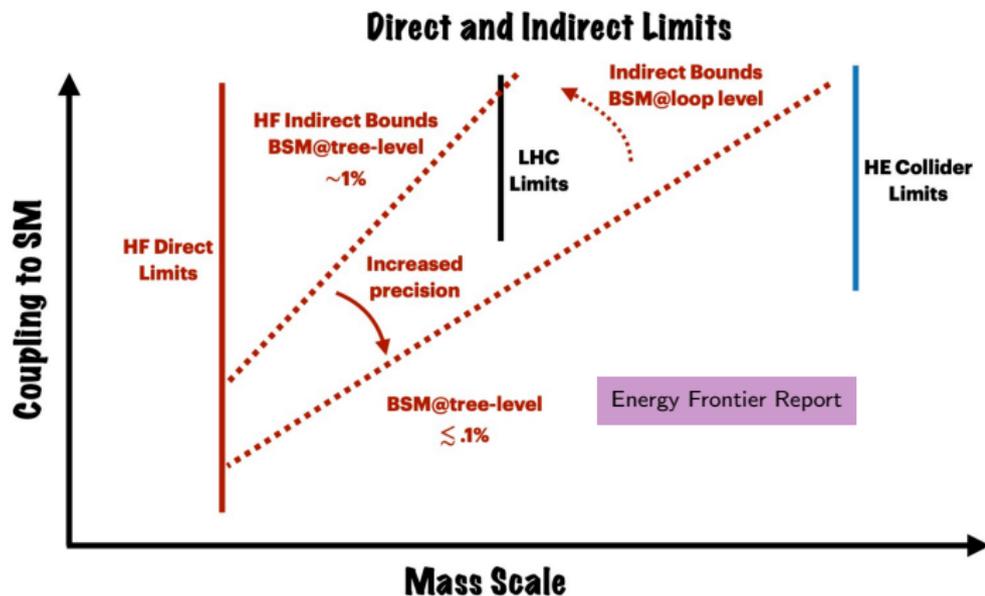
Slide inspired by G. Salam / L. Cieri...

(Better) Theory Uncertainties



- Principle of Maximum Conformality, Di Giustino, Brodsky, Wang Wu [[arXiv:2002.01789](https://arxiv.org/abs/2002.01789)]
- Probabilistic definition of the perturbative theoretical uncertainty, Bonvini [[arXiv:2006.16293](https://arxiv.org/abs/2006.16293)]
- Bayesian estimates for missing higher orders in perturbative calculations, Duhr, Huss, Mazeliauskas, Szafron [[arXiv:2106.04585](https://arxiv.org/abs/2106.04585)]

Squeezing the physics from collider data



Introduction

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Five+ Scales at Two Loops

| | Integrals | Amplitudes (analytic) |
|------|---|---|
| 2015 | 5pt 0M P, Papadopoulos et al. [arXiv:1511.09404] | 5 gluon all-plus LC, Gehrmann et al. [arXiv:1511.05409] |
| 2016 | | |
| 2017 | | |
| 2018 | 5pt 0M P (pent funcs), Gehrmann et al. [arXiv:1807.09812] | 5 gluon single-minus LC, Badger et al. [arXiv:1811.11699] 5 gluon LC, Abreu et al. [arXiv:1812.04586] |
| 2019 | 5pt 1M NP HB, Papadopoulos et al. [arXiv:1910.06275] | 5 parton LC, Abreu et al. [arXiv:1904.00945] 5 gluon all-plus, Badger et al. [arXiv:1905.03733] |
| 2020 | 5pt 1M P, Abreu et al. [arXiv:2005.04195] 5pt 1M P, Canko et al. [arXiv:2009.13917] 5pt 0M NP (pent funcs), Chicherin et al. [arXiv:2009.07803] | 2-q 3- γ LC, Abreu et al. [arXiv:2010.15834], Chawdhry et al. [arXiv:2012.13553] 3-p 2- γ LC, Agarwal et al. [arXiv:2102.01820], Chawdhry et al. [arXiv:2103.04319] 3 jet LC, Abreu et al. [arXiv:2102.13609] 3-p 2- γ , Agarwal et al. [arXiv:2105.04585] 2-q H 2-b LC, Badger et al. [arXiv:2107.14733] 4-p 2-l LC, Abreu et al. [arXiv:2110.07541] |
| 2021 | 5pt 1M NP HB, Abreu et al. [arXiv:2107.14180] 5pt 1M P (pent funcs), Chicherin et al. [arXiv:2110.10111] | 5pt 1M NP HB, Kardos et al. [arXiv:2201.07509] 3-p γ 2-l LC, Badger et al. [arXiv:2201.04075] |

Four-Point at Three loops

By now all **three-loop four-parton** and **two-parton two-photon** amplitudes have been computed

- Integrals

- **4-point massless**, Henn, Mistlberger, Smirnov, Wasser [[arXiv:2002.09492](#)]
- **4-point 1-mass tennis-court**, Canko, Syrrakos [[arXiv:2112.14275](#)]

- Amplitudes

- **2-quark 2- γ** , Caola, von Manteuffel, Tancredi [[arXiv:2011.13946](#)]
- **4-quark**, Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi [[arXiv:2108.00055](#)]
- **2-gluon 2- γ** , Bargiela, Caola, von Manteuffel, Tancredi [[arXiv:2111.13595](#)]
- **4-gluon**, Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi [[arXiv:2112.11097](#)]
- **2-quark 2-gluon**, Caola, Chakraborty, Gambuti, von Manteuffel, Tancredi [[arXiv:2207.03503](#)]

Related Progress at Four+ Loops

- **Four-loop form factors for $2 \rightarrow 1$ processes**
 - Henn, Smirnov, Smirnov, Steinhauser [[arXiv:1604.03126](#)]
 - Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser [[arXiv:2202.04660](#)]
 - Chakraborty, Huber, Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser [[arXiv:2204.02422](#)]
- **Progress on four-loop splitting functions**
 - Moch, Ruijl, Ueda, Vermaseren, Vogt [[arXiv:1707.08315](#)]
 - Moch, Ruijl, Ueda, Vermaseren, Vogt [[arXiv:2111.15561](#)]
 - (See the recent usage in the “ aN^3LO ” set from the MSHT PDF set! [[arXiv:2207.04739](#)])
- **Five-loop beta functions**
 - Herzog, Ruijl, Ueda, Vermaseren, Vogt [[arXiv:1701.01404](#)]
 - Luthe, Maier, Marquard, Schroder [[arXiv:1709.07718](#)]

The Future: Immediate Needs

Summary of the **Les Houches precision wishlist** for hadron colliders.

HTL stands for calculations in heavy top limit, VBF* stands for structure function approximation

| process | known | desired |
|--------------------------------------|---|---|
| $pp \rightarrow H$ | $N^3\text{LO}_{\text{HTL}}, N^2\text{LO}_{\text{QCD}}^{(1)}, N^{(1,1)}\text{LO}_{\text{QCD}\otimes\text{EW}}^{(\text{HTL})}$ | $N^4\text{LO}_{\text{HTL}} \text{ (incl.)}, N^3\text{LO}_{\text{QCD}}^{(\text{SC})}$ |
| $pp \rightarrow H + j$ | $N^3\text{LO}_{\text{HTL}}, \text{NLO}_{\text{QCD}}, N^{(1,1)}\text{LO}_{\text{QCD}\otimes\text{EW}}$ | $N^4\text{LO}_{\text{HTL}} \otimes \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow \bar{H} + 2j$ | $\text{NLO}_{\text{HTL}} \otimes \text{LO}_{\text{QCD}}$ $N^3\text{LO}_{\text{QCD}}^{(\text{VBF}^*)} \text{ (incl.)}, N^2\text{LO}_{\text{QCD}}^{(\text{VBF}^*)}, \text{NLO}_{\text{EW}}^{(\text{VBF}^*)}$ | $N^4\text{LO}_{\text{HTL}} \otimes \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}},$ $N^2\text{LO}_{\text{QCD}}^{(\text{VBF}^*)}$ |
| $pp \rightarrow H + 3j$ | $\text{NLO}_{\text{HTL}}, \text{NLO}_{\text{QCD}}^{(\text{VBF}^*)}$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow VH$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}, \text{NLO}_{gg \rightarrow HZ}^{(\text{L})}$ | |
| $pp \rightarrow VH + j$ | $N^2\text{LO}_{\text{QCD}}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow HH$ | $N^3\text{LO}_{\text{HTL}} \otimes \text{NLO}_{\text{QCD}}$ | NLO_{EW} |
| $pp \rightarrow H + t\bar{t}$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}, N^2\text{LO}_{\text{QCD}} \text{ (off-diag.)}$ | $N^2\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow H + t/\bar{t}$ | NLO_{QCD} | $N^2\text{LO}_{\text{QCD}}, \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow V$ | $N^3\text{LO}_{\text{QCD}}, N^{(1,1)}\text{LO}_{\text{QCD}\otimes\text{EW}}, \text{NLO}_{\text{EW}}$ | $N^4\text{LO}_{\text{QCD}} + N^{(1,1)}\text{LO}_{\text{QCD}\otimes\text{EW}}, N^2\text{LO}_{\text{EW}}$ |
| $pp \rightarrow VV'$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}, + \text{NLO}_{\text{QCD}} (gg)$ | $\text{NLO}_{\text{QCD}} (gg, \text{massive loops})$ |
| $pp \rightarrow V + j$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | hadronic decays |
| $pp \rightarrow V + 2j$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}, \text{NLO}_{\text{EW}}$ | $N^2\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow V + b\bar{b}$ | NLO_{QCD} | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow VV' + 1j$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | $N^2\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow VV' + 2j$ | $\text{NLO}_{\text{QCD}} (\text{QCD}), \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} (\text{EW})$ | Full $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow W^+W^+ + 2j$ | Full $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | |
| $pp \rightarrow W^+W^- + 2j$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} (\text{EW component})$ | |
| $pp \rightarrow W^+Z + 2j$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} (\text{EW component})$ | |
| $pp \rightarrow ZZ + 2j$ | Full $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | |
| $pp \rightarrow VV'V''$ | $\text{NLO}_{\text{QCD}}, \text{NLO}_{\text{EW}} \text{ (w/o decays)}$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow W^+W^+W^-$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | |
| $pp \rightarrow \gamma\gamma$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | $N^3\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow \gamma + j$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | $N^3\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow \gamma\gamma + j$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}, + \text{NLO}_{\text{QCD}} (gg \text{ channel})$ | |
| $pp \rightarrow \gamma\gamma\gamma$ | $N^2\text{LO}_{\text{QCD}}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow 2\text{jets}$ | $N^3\text{LO}_{\text{QCD}}, \text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | $N^3\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ |
| $pp \rightarrow 3\text{jets}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}}$ | |
| $pp \rightarrow \bar{t}\bar{t}$ | $N^2\text{LO}_{\text{QCD}} \text{ (w/ decays)} + \text{NLO}_{\text{EW}} \text{ (w/o decays)}$ $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays, off-shell effects)}$ $N^3\text{LO}_{\text{QCD}}$ | $N^3\text{LO}_{\text{QCD}}$ |
| $pp \rightarrow \bar{t}\bar{t} + j$ | $\text{NLO}_{\text{QCD}} \text{ (w/ decays, off-shell effects)}$ $\text{NLO}_{\text{EW}} \text{ (w/o decays)}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ |
| $pp \rightarrow \bar{t}\bar{t} + 2j$ | $\text{NLO}_{\text{QCD}} \text{ (w/o decays)}$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ |
| $pp \rightarrow \bar{t}\bar{t} + Z$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/o decays)}$ $\text{NLO}_{\text{QCD}} \text{ (w/ decays, off-shell effects)}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ |
| $pp \rightarrow \bar{t}\bar{t} + W$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays, off-shell effects)}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ |
| $pp \rightarrow t/\bar{t}$ | $N^2\text{LO}_{\text{QCD}}^*(\text{w/ decays})$ $\text{NLO}_{\text{EW}} \text{ (w/o decays)}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ |
| $pp \rightarrow tZj$ | $\text{NLO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/ decays)}$ | $N^2\text{LO}_{\text{QCD}} + \text{NLO}_{\text{EW}} \text{ (w/o decays)}$ |

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Computing Scattering Amplitudes

$$\mathcal{A} = \sum_{\kappa} D_{\kappa} \xrightarrow[\text{Projectors}]{\text{Feynman diags}} \sum_i c_i F_i \xrightarrow[\text{IBPs}]{\text{Form factors}} \sum_j C_j \mathcal{I}_j \xrightarrow[\text{UV/IR renorm}]{\text{Master ints}} \sum_k r_k h_k \xrightarrow[\text{simplifications}]{\text{Special functs}}$$

Many recent advances make possible recent progress: more optimal techniques for different steps or as shortcuts

Computing Scattering Amplitudes

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Many recent advances make possible recent progress: more optimal techniques for different steps or as shortcuts

- Usage of **method of differential equations** for integration spread out, in particular in *canonical form* [arXiv:1304.1806]
- Deeper understanding of **function spaces** [arXiv:2203.07088] allow for analytic expressions and more efficient numerical evaluations
- Numerical approaches based on **sector decomp** automated in public codes pySecDec and Fiesta
- Building **finite bases** of master integrals have been automated (see e.g. [arXiv:1701.06583])
- Numerically solving diff equations through **generalized series expansions** [arXiv:1907.13234] has gained momentum, with public implementations appearing (e.g. DiffExp)
- Promising technique for boundary values: the **auxiliary mass flow method** [arXiv:2107.01864], implemented in the package AMFlow

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- **Advanced one-loop tools** available in programs like Helac-NLO, MG5_aMC@NLO, NLOX, OpenLoops, Recola
- Multi-loop analytic integrands aided by better **projector methods** [arXiv:1906.03298]
- **Numerical unitarity** extended to two-loop amplitudes [arXiv:1703.05273], exploiting a novel *master-surface* integrand parametrization [arXiv:1510.05626]
- Major advances in **tools for IBP reduction**, like for example Fire, Reduze, LiteRed and Kira
- Methods based on numerical evaluations in **finite fields** and **functional reconstruction** [arXiv:1406.4513] [arXiv:1608.01902] have become standard
- Many advances in **simplifications** of complex expressions, e.g. by developing multivariate partial fraction algorithms [arXiv:1904.00945]

Two-Loop Numerical Unitarity

Decompose \mathcal{A} in terms of *master integrals*:

$$\mathcal{A}^{(L)} = \sum_{\Gamma \in \Delta} \sum_{i \in M_{\Gamma}} c_{\Gamma,i} \mathcal{I}_{\Gamma,i}$$

Drop the integral symbol, introducing the *integrand ansatz*:

$$\mathcal{A}^{(L)}(\ell_l) = \sum_{\Gamma \in \Delta} \sum_{k \in Q_{\Gamma}} c_{\Gamma,k} \frac{m_{\Gamma,k}(\ell_l)}{\prod_{j \in P_{\Gamma}} \rho_j(\ell_l)}$$

Functions $Q_{\Gamma} = \{m_{\Gamma,k}(\ell_l) | k \in Q_{\Gamma}\}$ *parametrize* every possible integrand (up to a given power of loop momenta). *E.g.:*

- **Tensor Basis:** construct Q from *monomials of loop momenta* (parameters). Easy to build for general integrands, non-trivial relation to master integrals. Easy to extract function-space dim
- **Master-Surface Basis:** a clever choice of parametrization makes mapping to master integrals straightforward [*Ita, arXiv:1510.05626*]. Break $Q_{\Gamma} = M_{\Gamma} \cup S_{\Gamma}$, where S_{Γ} *integrate to zero* and M_{Γ} *correspond to master integrands*

Consider the **integration by parts (IBP)** relation on Γ

$$0 = \int \prod_i d^D \ell_i \frac{\partial}{\partial \ell_j^\nu} \left[\frac{u_j^\nu}{\prod_{k \in P_\Gamma} \rho_k} \right]$$

making it *unitarity compatible* (controlling the **propagator structure**) [Gluza, Kadja, Kosower '10; Schabinger '11]

$$u_j^\nu \frac{\partial}{\partial \ell_j^\nu} \rho_k = f_k \rho_k$$

Write ansatz for u_j^ν expanded in external and loop momenta, and find solution to the polynomial equations using the CAS **SINGULAR**

Build a full set of surface terms and fill the rest of the space with **master integrands**

Related [Boehm, Georgoudis, Larsen, Schulze, Zhang '16 - '19]
[Agarwal, von Manteuffel '19]

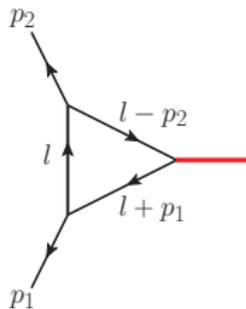
A 1-loop Example for Surface Terms: Part 1

Consider the 1-loop 1-mass triangle with

$$\rho_1 = (\ell + p_1)^2, \quad \rho_2 = \ell^2, \quad \rho_3 = (\ell - p_2)^2$$

and we construct $u^\nu \partial / \partial \ell^\nu$ by parametrizing

$$u^\nu = u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu$$



We then get the **syzygy equation** (polynomial equation):

$$(u_1^{\text{ext}} p_1^\nu + u_2^{\text{ext}} p_2^\nu + u^{\text{loop}} \ell^\nu) \frac{\partial}{\partial \ell^\nu} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix} - \begin{pmatrix} f_1 \rho_1 \\ f_2 \rho_2 \\ f_3 \rho_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can then show that we have the solution for the **IBP-generating vector**:

$$u^\nu \frac{\partial}{\partial \ell^\nu} = [(\rho_3 - \rho_2) p_1^\nu + (\rho_1 + \rho_2) p_2^\nu + (-s + 2\rho_3 - 2\rho_2) \ell^\nu] \frac{\partial}{\partial \ell^\nu}$$

A 1-loop Example for Surface Terms: Part 2

Now we have the surface term:

$$0 = \int d^D \ell \frac{\partial}{\partial l^\nu} \frac{u^\nu}{\rho_1 \rho_2 \rho_3} = \int d^D \ell \frac{1}{\rho_1 \rho_2 \rho_3} [-(D-4)s - 2(D-3)\rho_2 + 2(D-3)\rho_3]$$

The scalar triangle integrand **can be replaced by a surface term**, though commonly it is kept, the corresponding “master” integral in OPP reduction.

The IBP relation between the triangle and the $s = (p_1 + p_2)^2$ bubble is:

$$-(D-4)sI_{\text{tri}} - 2(D-3)I_{\text{s-bub}} = 0$$

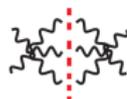
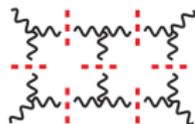
Similar manipulations can be carried out at **two loops**. More complicated **syzygy equations** (polynomial relations) need to be solved (e.g. with **SINGULAR**)

Unitarity Approach to Computing Integrand Coefficients

[Bern, Dixon, Dunbar, Kosower] [Britto, Cachazo, Feng]

- In **on-shell configurations** of ℓ_l , the integrand factorizes

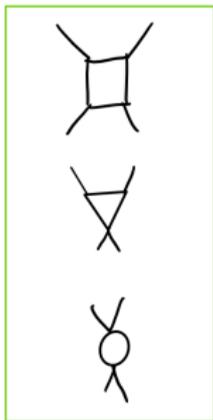
$$\sum_{\text{states}} \prod_{i \in T_\Gamma} \mathcal{A}_i^{\text{tree}}(\ell_l^\Gamma) = \sum_{\substack{\Gamma' \geq \Gamma \\ k \in \bar{Q}_{\Gamma'}}} \frac{c_{\Gamma',k} m_{\Gamma',k}(\ell_l^\Gamma)}{\prod_{j \in (P_{\Gamma'} / P_\Gamma)} \rho_j(\ell_l^\Gamma)}$$



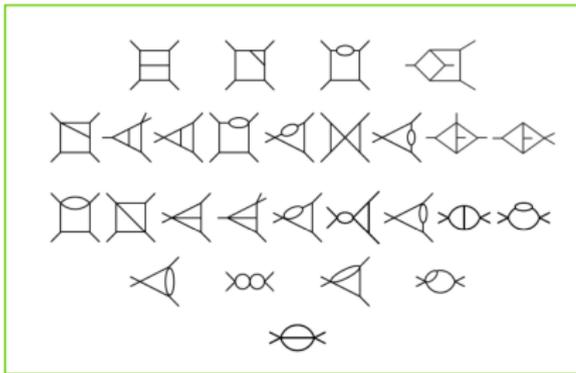
- Need **efficient computation** of (products of) **tree-level amplitudes**
 - On-shell recursions, Berends-Giele relations, etc
 - D_s -dimensional **state sum**
- **Never construct** analytic integrand, numerics for every phase-space point!

NUMERICAL STABILITY:

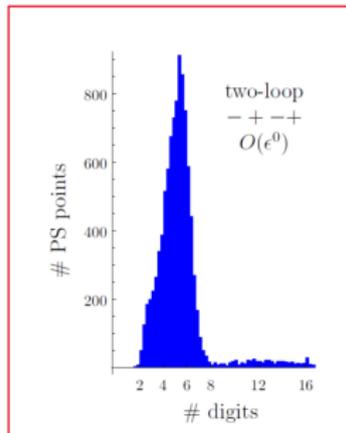
e.g. 4-gluon amplitudes



Function spaces with $\mathcal{O}(10/50)$ dimensions



Function spaces with $\mathcal{O}(100/1000)$ dimensions

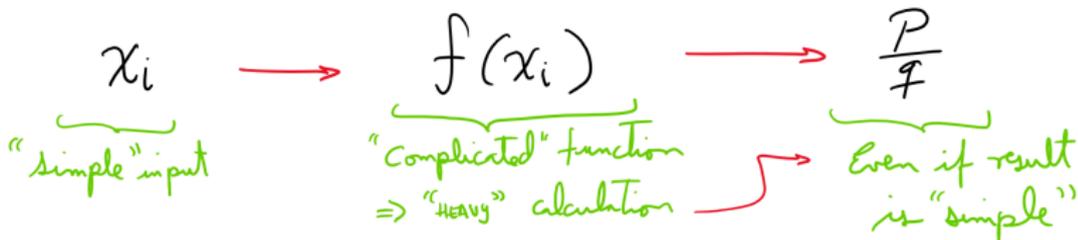


- * Relative precision of two-loop 4-gluon amp numerical calculation
- * High-precision floating point arithmetic a remedy

[Abreu, FFC, Ita, Jaquier, Page, Zeng, '17]

MODULAR ALGEBRA: [von Manteuffel, Schabinger, 2014]

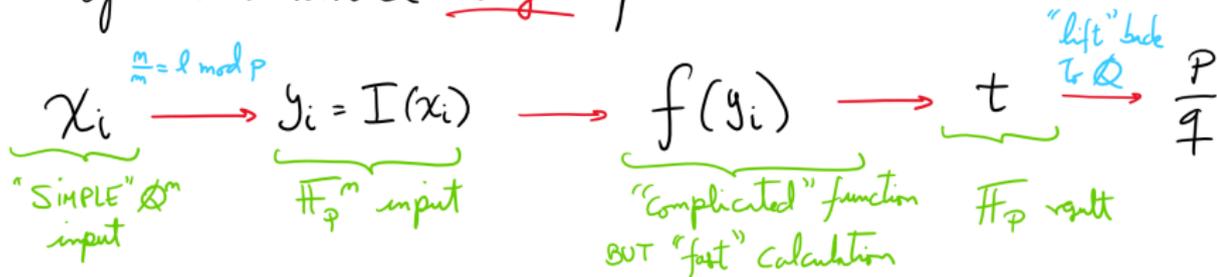
- * Integral reduction can be performed **exactly** in CAS if kinematical info is **RATIONAL** ($x_i \in \mathbb{Q}^m$)
- * Nevertheless, **RATIONAL** computer algebra reflects the numerical complexity of corresponding **ANALYTIC STRUCTURE (COMPUTATIONAL ALGORITHM)**



FINITE (NUMBER) FIELDS: [von Manteuffel, Schabinger, 2014]

* MAP \mathbb{Q}^m into \mathbb{F}_p^m and try to reconstruct result!

* If **cardinality** p is smaller than CPU's word size (2^{64}) operations will be very fast



* "Lift" back operation, or **rational reconstruction** works well if $\frac{p}{q}$ is "simple" enough (OR MORE \mathbb{F}_p 's needed!).

Extracting Functional Form from Numerics

INTEGRAL COEFFS AS FUNCTIONS of ϵ :

(INTEGRAND'S ANSATZ)

$$A(l_e) = \sum_{\Gamma, i} C_{\Gamma, i} \frac{m_{\Gamma, i}(l_e)}{\prod_{k \in \Gamma} \beta_k(l_e)} \rightarrow C_{\Gamma, i} \text{ are functions of } x_k \text{ \& } D=4-2\epsilon$$

Indeed $C_{\Gamma, i}$ appears as rational functions of ϵ

$$C_{\Gamma, i} = \frac{\sum_j f_j(x_k) \epsilon^{j+N}}{\sum_j g_j \epsilon^{j+M}}$$

} STRUCTURE NOT KNOWN & PRIORI!

ϵ dependence comes from the structure of $m_{\Gamma, i}(l_e)$ and through linear algebra ("subtraction" procedure)

Functional Reconstruction from Numeric Samples

THIELE'S INTERPOLATION FORMULA:

Every rational function can be written as a *continued fraction*

$$f(x) = \frac{\sum_{r=0}^R n_r x^r}{\sum_{r'=0}^{R'} d_{r'} x^{r'}} = a_0 + \frac{x - y_0}{a_1 + \frac{x - y_1}{a_2 + \frac{x - y_2}{\dots + \frac{x - y_{N-1}}{a_N}}}}$$

- * Determine a_i by *evaluating* $f(y_i)$ (y_i random)
- * Stop when $f(y_{i+1})$ *matches* interpolated value (+ *estim check*)
- * Through only *field operations* recover rational function
(FF's result can be lifted to \mathbb{Q})

See also [Peraro, arXiv:1608.01902] for multi-variate case
[Peraro, FiniteFlow, arXiv:1905.08019]
[Klappert, Klein, Lange, Firefly, arXiv:2004.01463]

Removing Lower-Order Information

IR structure: $A_R^{(1)} = \underbrace{I^{(1)}}_{\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon} \text{ poles}} A_R^{(0)} + \mathcal{O}(\varepsilon^0)$

$$A_R^{(2)} = \underbrace{I^{(1)} A_R^{(1)} + I^{(2)} A_R^{(0)}}_{\frac{1}{\varepsilon^4}, \dots, \frac{1}{\varepsilon} \text{ poles}} + \mathcal{O}(\varepsilon^0)$$

Define Remainders: $R^{(1)} = A_R^{(1)} - I^{(1)} A_R^{(0)} + \mathcal{O}(\varepsilon)$

$$R^{(2)} = A_R^{(2)} - I^{(1)} A_R^{(1)} - I^{(2)} A_R^{(0)} + \mathcal{O}(\varepsilon)$$

Optimize Ansatz

By physical constraints:

$$r^{\pm}(s_{ij}) = \frac{\eta^{\pm}(s_{ij})}{\prod_k W_k^{\alpha_k}(s_{ij})}$$

↗ Polynomial

↘ Special function's arguments (Alphabet letter)

Determining $\prod_k W_k^{\alpha_k}(s_{ij})$ can be achieved by *univariate* reconstruction in curve $s_{ij}(\lambda)$ and polynomial division!

Multivariate reconstruction reduced to determination of the polynomials $\eta_k^{\pm}(s_{ij})$

→ Simplify by multivariate partial fractions! RELATED TO MULTIVARIATE PART [Heller, von MANTOUFFEL]

Introduction

State-of-the-art and future needs

A touch on techniques

Our survey & outlook

Survey on Multi-Loop Developments for Colliders

- As part of our white paper [[arXiv:2204.04200](https://arxiv.org/abs/2204.04200)] we performed a survey about resources needed to complete recent state-of-the-art calculations for precision collider phenomenology
- We received information about calculations appearing in 53 scientific publications
- Example questions:
 - What computational resources did you employ?
 - How many PhD/PD years went into this project?
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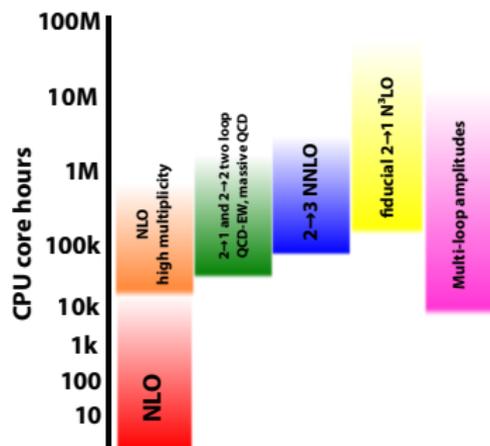
We thank them all as well as their collaborators! Samuel Abreu, Bakul Agarwal, Konstantin Asteriadis, Simon Badger, Matteo Becchetti, Marco Bonetti, Federico Buccioni, Luca Buonocore, Fabrizio Caola, Gudrun Heinrich, Alexander Huss, Stephen P. Jones, Stefan Kallweit, Matthias Kerner, Matteo Marcoli, Javier Mazzitelli, Johannes Michel, Sven Moch, Marco Niggetiedt, Costas Papadopoulos, Mathieu Pellen, Rene Poncelet, Jérémie Quarroz, Luca Rottoli, Gabor Somogyi, Qian Song, Vasily Sotnikov, Matthias Steinhauser, Gherardo Vita, Chen-Yu Wang, Stefan Weinzierl, Marius Wiesemann, Malgorzata Worek, Tongzhi Yang and YuJiao Zhu.

Some of the Feedback

- HPC usage a standard in our community. For the first time HPC systems used for CAS!
- Typical project requires 2-5 PhD/PD years to complete, and often rely on a decade (or more) of developments
- Numerical and semi-numerical methods on the forefront, we forecast significant rise. GPU usage not spread out

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- Estimate of HPC usage:



Outlook

- Settling the SM status at the (under) 1% level will be one great achievement of the LHC, and we look forward to even more!
- After this Snowmass cycle we might expect as common place matched $2 \rightarrow 3$ NNLO studies, fixed order $2 \rightarrow 2$ N³LO calculations and even N⁴LO results
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Thanks!